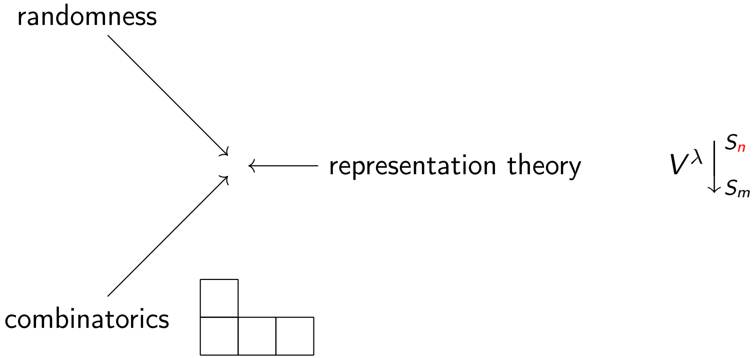
museum of visual ART
Asymptotic Representation Theory

guided tour with Piotr Śniady

transparencies, references, homework available on
`http://psniady.impan.pl/fpsac`

`Version: 2022-06-23 19:55:42+02:00`

one-slide summary



visual viewpoint on algebraic combinatorics creates nice questions

one-slide summary

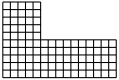
randomness

asymptotic $n \to \infty$

$\longleftarrow$ representation theory

$V^{\lambda} \Big\downarrow^{S_n}_{S_m}$

combinatorics



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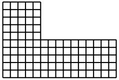


visual

visual viewpoint on algebraic combinatorics creates nice questions

plan for today

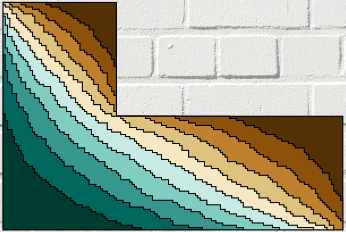


exhibit A

what can you say
about random Young diagrams?

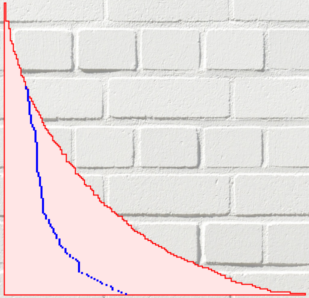
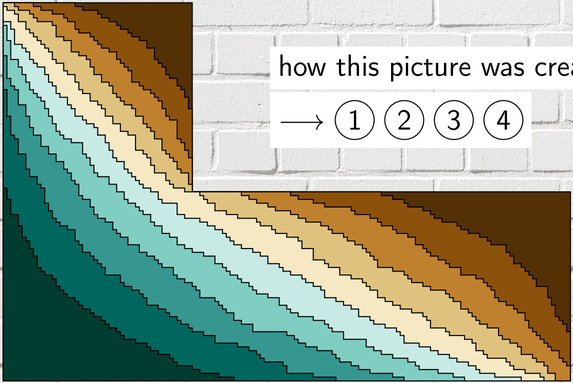


exhibit B

what can you say
about RSK
applied to random input?

exhibit A



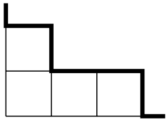
how this picture was created?

→ (1) (2) (3) (4)

what can we say

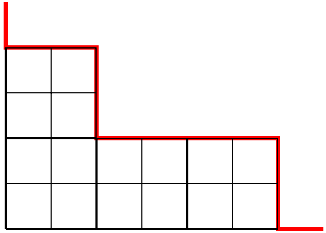
about random Young diagrams and random Young tableaux?

① scaling



ξ

start with a Young diagram ξ...



$\lambda = s\xi$

...and scale it by a factor $s \in \{1, 2, \dots\}$

---

if $s > 0$ is a real number, $s\xi$ is a *generalized Young diagram*

$\Lambda = \frac{1}{\sqrt{|\lambda|}}\lambda$ is called *asymptotic shape of* λ

② select randomly a standard tableau with shape λ

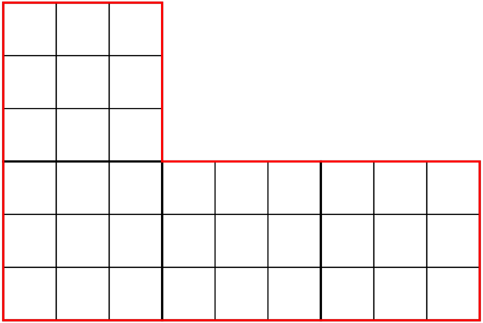


diagram λ

② select randomly a standard tableau with shape λ

20	23	31						
9	15	30						
8	14	28						
6	7	12	17	22	26	29	33	36
3	4	11	13	18	21	25	32	35
1	2	5	10	16	19	24	27	34

tableau T

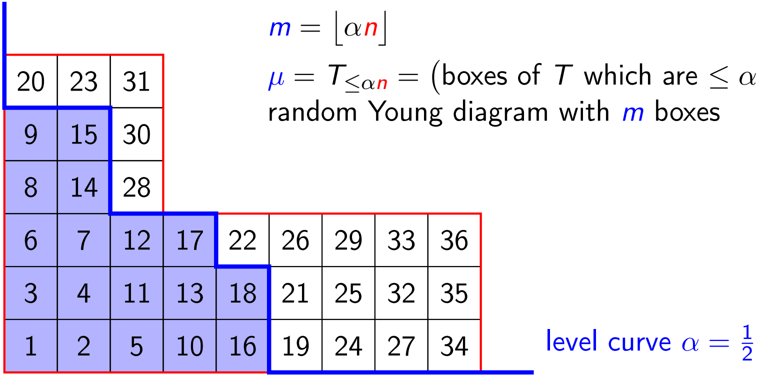
③ draw the level curves

fix a real number $0 \leq \alpha \leq 1$

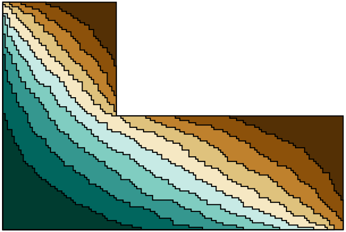
$n = |\lambda|$

$m = \lfloor \alpha n \rfloor$

$\mu = T_{\leq \alpha n} = \big(\text{boxes of } T \text{ which are } \leq \alpha n\big)$ is a random Young diagram with m boxes



④ this is a layer tinting of a random standard tableau of fixed shape $\color{red}{\lambda}$



hint: $\color{red}{\lambda} = s\xi$

for best viewing experience

scale the picture by $\frac{1}{s}$

then $s \to \infty$

reducible representation ⟶ random Young diagram μ

let W be a reducible representation of the symmetric group S_m;
its decomposition into irreducibles is given by

$$W = \bigoplus_{\mu \vdash m} \mathfrak{m}_\mu V^\mu$$

$\mathfrak{m}_\mu \in \{0, 1, \dots\}$ is the multiplicity of V^μ in W

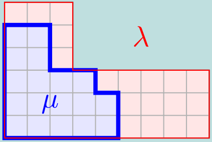
we declare the probability of sampling the Young diagram μ to be equal to

$$\mathbb{P}_W(\mu) = \frac{\mathfrak{m}_\mu \dim V^\mu}{\dim W}$$

"random irreducible component of W"

our concrete example

$$W = V^\lambda \big\downarrow^{S_n}_{S_m}$$



framework

cycles

$$[k] = (1, \dots, k) \quad \in S_m \qquad \text{for } m \geq k,$$
$$[k, \ell] = (1, \dots, k)(k+1, \dots, k+\ell) \quad \in S_m \qquad \text{for } m \geq k+\ell$$

---

characters

$$\chi_W(\pi) = \frac{\operatorname{Tr} \rho_W(\pi)}{\operatorname{Tr} \rho_W(\mathrm{id})} \quad \text{for } \pi \in S_m$$

---

asymptotic setting

somebody gives us some interesting sequence $W_1, W_2, \dots$
W_m is a (reducible or irreducible) representation of S_m,

$\chi_m = \chi_{W_m}$ is its character

$\lambda^{(m)}$ is a random Young diagram with m boxes,
the random irreducible component of W_m

limit shape $\longleftrightarrow$ characters

the following two conditions are equivalent
(if you add sufficiently many technical assumptions):

the asymptotic shape $\frac{1}{\sqrt{m}}\lambda^{(m)}$ converges to some limit shape Λ

the character χ_m has a specific asymptotic behavior
which depends on the limit shape Λ

$\longrightarrow$Philippe Biane 1998, 2001

limit shape $\longrightarrow$ characters

start with a Young diagram ξ

$\lambda^{(n)} := s\xi$ if n is of the form $n = s^2|\xi|$,

in this way $\frac{1}{\sqrt{n}}\lambda^{(n)} = \frac{1}{\sqrt{|\xi|}}\xi = \Lambda$,

$W_n = V^{\lambda^{(n)}}$

Theorem (Philippe Biane 1998)

for each $k \in \{1, 2, \dots\}$ the limit

$$R_{k+1}(\Lambda) := \lim_{n\to\infty} \chi_n([k])\, n^{\frac{k-1}{2}}$$

exists and is a nice function of the limit shape Λ

$R_2, R_3, \dots$ are called free cumulants of Λ

characters ⟶ limit shape

- assume that for each $k \in \{1, 2, \dots\}$ the limit exists

$$R_{k+1} := \lim_{m\to\infty} \chi_m\big([k]\big)\, m^{\frac{k-1}{2}}$$

- assume that

$$\chi_m\big([k,l]\big) \approx \chi_m\big([k]\big)\chi_m\big([l]\big) \quad \text{for } m \to \infty;$$

let a random Young diagram $\mu^{(m)}$
be a random irreducible component of W_m

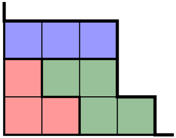
Theorem (Phlippe Biane 2001)

$$\frac{1}{\sqrt{m}}\mu^{(m)} \xrightarrow[m\to\infty]{\text{in probability}} \textit{generalized Young diagram}$$

with free cumulants $R_2, R_3, \dots$

exhibit A: why the limit curves exist?

why shape of $\lambda \longrightarrow$ character of V^λ?



classic tools:
Murnaghan–Nakayama rule

$\pi = [3, 4, 3]$

$\operatorname{Tr} \rho^\lambda(\pi) = (-1)^0 \cdot (-1)^1 \cdot (-1)^1 + \cdots$

lesson: old combinatorics is not useful today

can algebraic combinatorics
provide new exact formulas for characters
which are useful for asymptotic questions?

dual viewpoint on characters

for a Young diagram λ with n boxes
and $k \in \{1, 2, \dots\}$ we define

$$\mathrm{Ch}_k(\lambda) = \begin{cases} \underbrace{n \cdot (n-1) \cdots (n-k+1)}_{k \text{ factors}} \ \chi_\lambda\big([k]\big) & \text{if } n \geq k, \\ 0 & \text{if } n < k, \end{cases}$$

$\longrightarrow$ Ivanov, Kerov 1999

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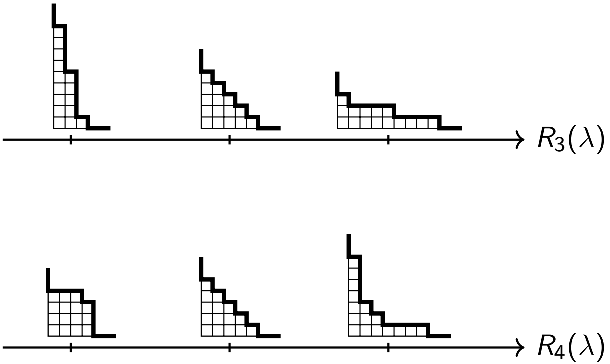
for each integer $k \geq 1$ and each Young diagram λ

$$\{1, 2, \dots\} \ni s \mapsto \mathrm{Ch}_k(s\lambda)$$

is a polynomial of degree $k+1$

free cumulants ⟷ shape

$$R_{k+1}(\lambda) = \left[s^{k+1}\right] \mathrm{Ch}_k(s\lambda) = \lim_{s\to\infty} \frac{1}{s^{k+1}} \mathrm{Ch}_k(s\lambda)$$



“asymptotic shape=free cumulants”

Kerov positivity conjecture

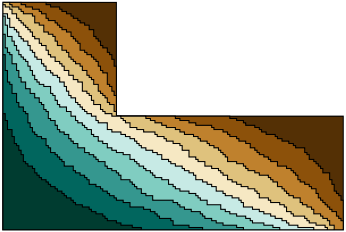
Biane's results are based on

$$\mathrm{Ch}_k \approx R_{k+1}$$

$$\begin{aligned}
\overbrace{\mathrm{Ch}_2}^{\text{character}} &= \overbrace{R_3}^{\text{shape}}, \\
\mathrm{Ch}_3 &= R_4 + R_2, \\
\mathrm{Ch}_4 &= R_5 + 5R_3, \\
\mathrm{Ch}_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2, \\
\mathrm{Ch}_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3
\end{aligned}$$

why positivity? $\longrightarrow$ Stanley–Féray character formula

exhibit A: moral lessons



- Biane's machinery has many more applications! homework ⟶ http://psniady.impan.pl/fpsac

- some classical tools of algebraic combinatorics are not convenient for asymptotic questions,
- asymptotic viewpoint may create new ("dual") tools in algebraic combinatorics,
- without asymptotic motivations you would not look for new character formulas,

outlook: Lassalle's conjecture

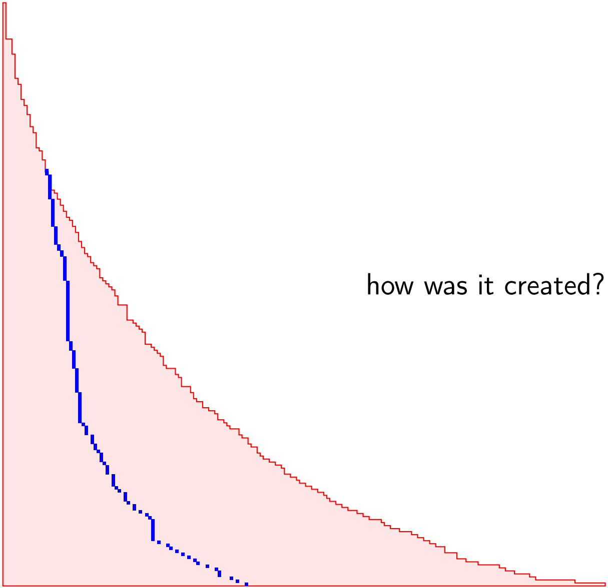
characters of the symmetric groups $\mathrm{Ch}_n =$ dual viewpoint on Schur polynomials

Jack characters $\mathrm{Ch}_k^{(\gamma)}$ = dual viewpoint on Jack polynomials (toy example of Macdonald polynomials)

$$\begin{aligned}
\mathrm{Ch}_1^{(\gamma)} &= R_2, \\
\mathrm{Ch}_2^{(\gamma)} &= R_3 + \gamma R_2, \\
\mathrm{Ch}_3^{(\gamma)} &= R_4 + 3\gamma R_3 + (1 + 2\gamma^2) R_2, \\
\mathrm{Ch}_4^{(\gamma)} &= R_5 + 6\gamma R_4 + \gamma R_2^2 + (5 + 11\gamma^2) R_3 + (7\gamma + 6\gamma^3) R_2,
\end{aligned}$$

why positivity?

exhibit B



how was it created? ⟶ RSK

Robinson–Schensted–Knuth algorithm is a bijection...

input:

- word $\mathbf{w} = (w_1, \dots, w_n)$

output:

- semistandard tableau P,
- standard tableau Q,

tableaux P and Q have the same shape with n boxes

---

example:

w = (23, 53, 74, 16, 99, 70, 82, 37, 41)

74	99		
23	53	70	
16	37	41	82

insertion tableau $P(\mathrm{w})$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathrm{w})$

Robinson-Schensted-Knuth algorithm — induction step

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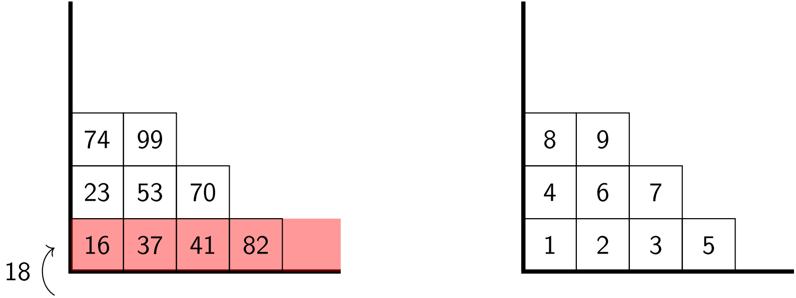

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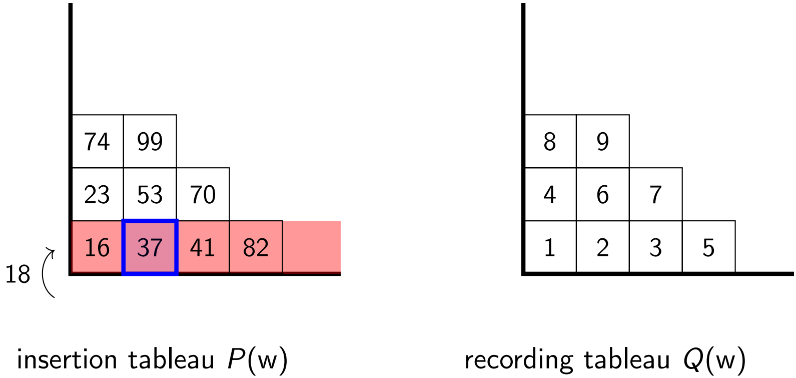

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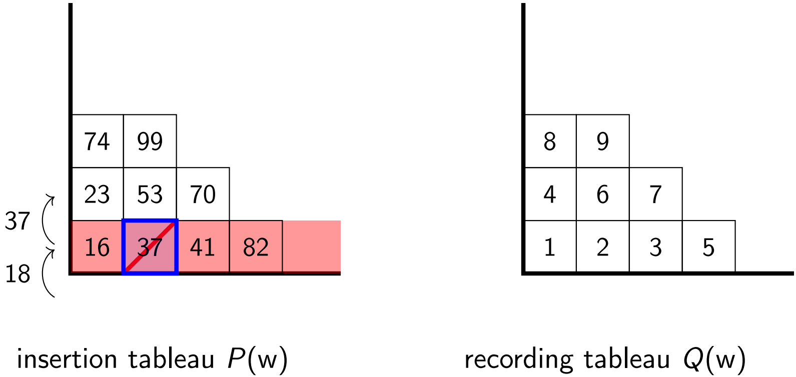

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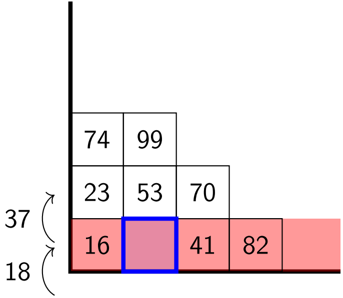
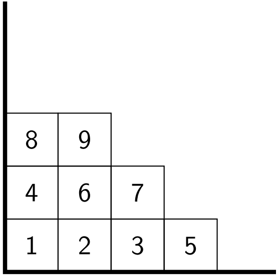


insertion tableau $P(\mathrm{w})$



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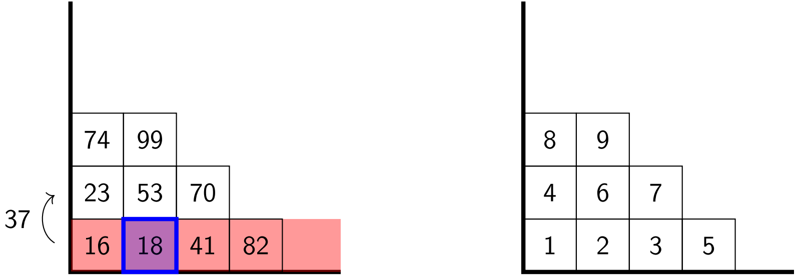

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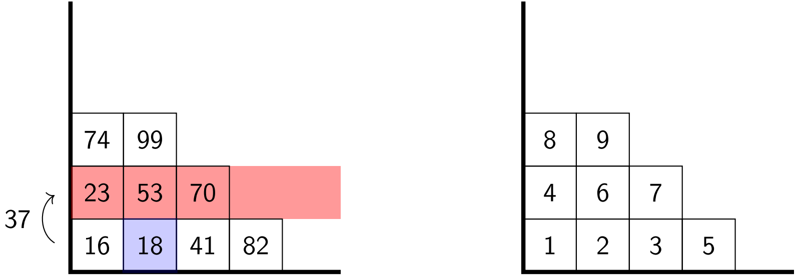

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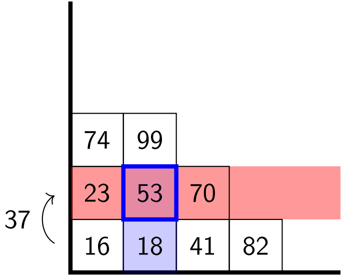
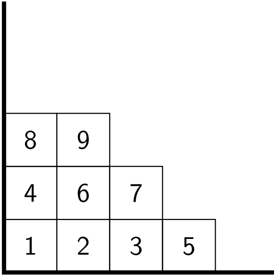


insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

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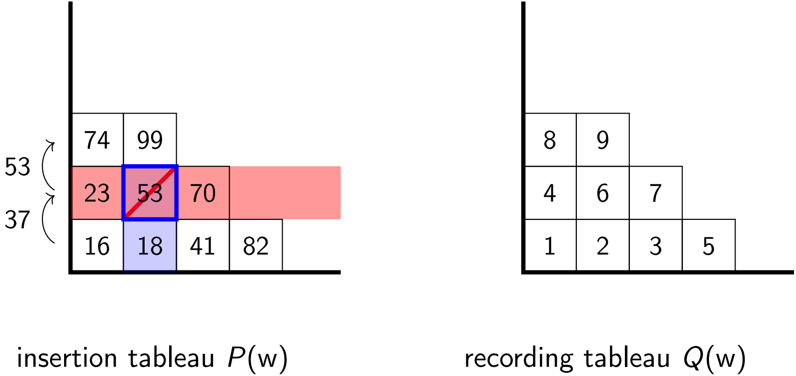

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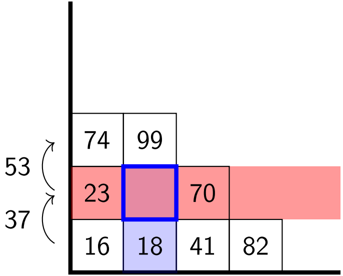

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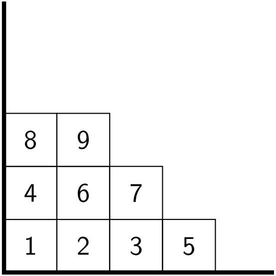

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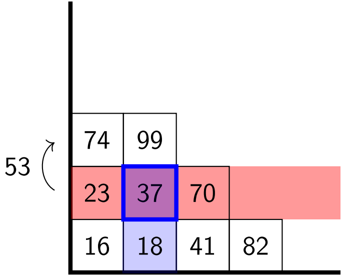
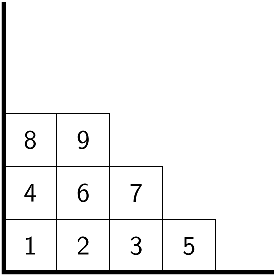


insertion tableau $P(w)$



recording tableau $Q(w)$

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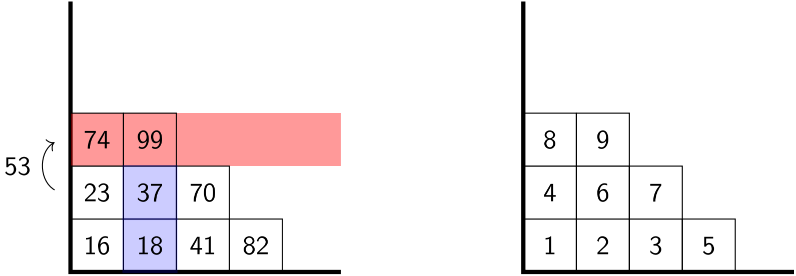

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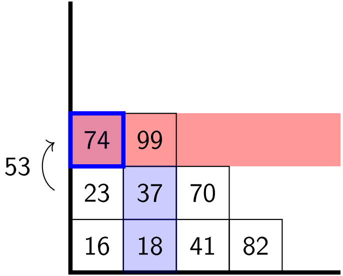

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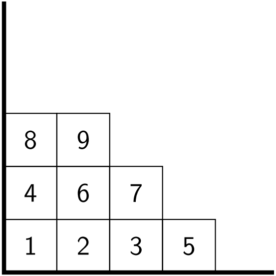

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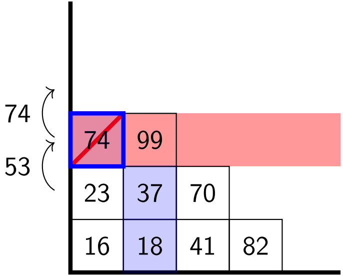

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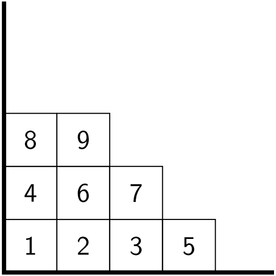

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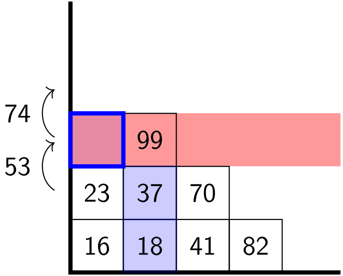
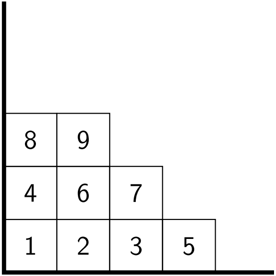


insertion tableau $P(\mathrm{w})$



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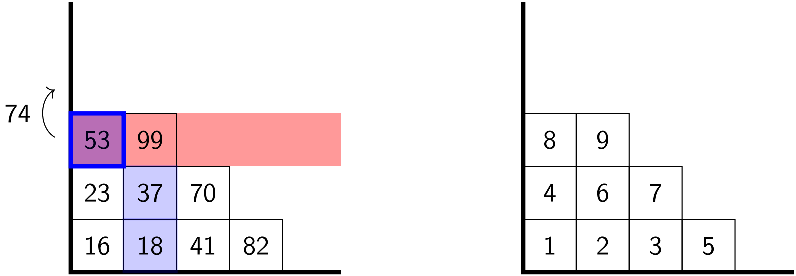

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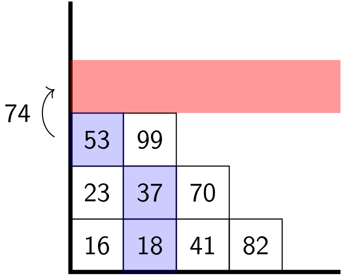
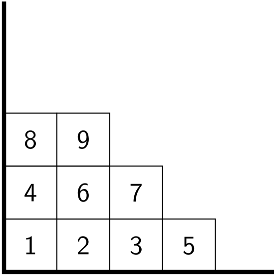
Robinson-Schensted-Knuth algorithm — induction step



insertion tableau $P(w)$



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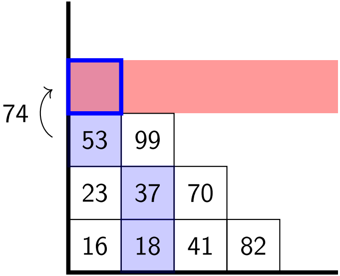
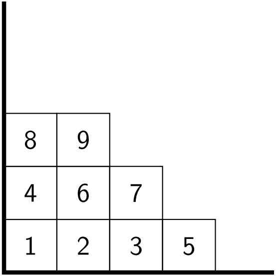
$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



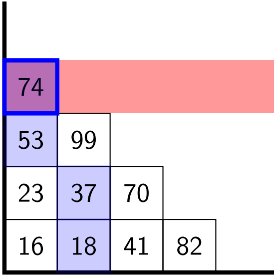
insertion tableau $P(\mathrm{w})$



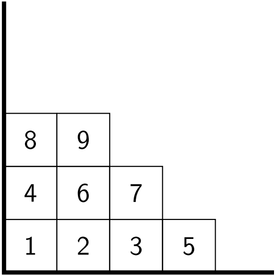
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



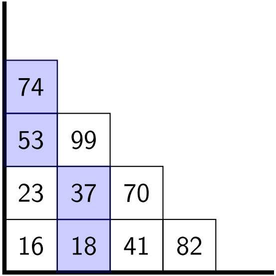
insertion tableau $P(\mathrm{w})$



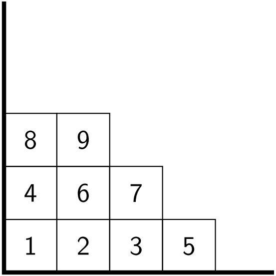
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



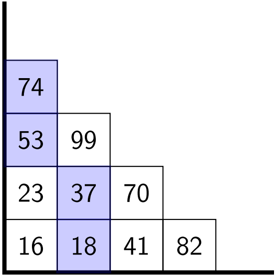
insertion tableau $P(\text{w})$



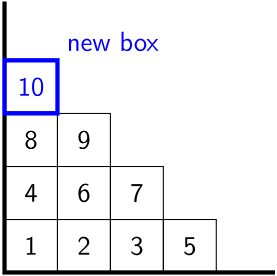
recording tableau $Q(\text{w})$

w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)

Robinson-Schensted-Knuth algorithm — induction step



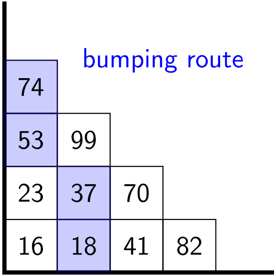
insertion tableau $P(w)$



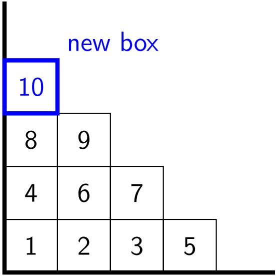
recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step

74			
53	99		
23	37	70	
16	18	41	82

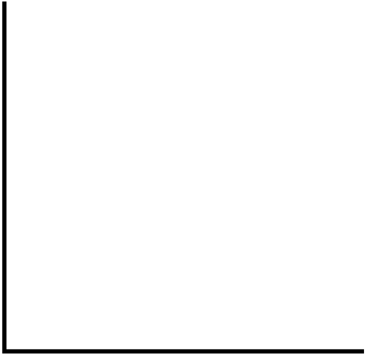
insertion tableau P(w)

10			
8	9		
4	6	7	
1	2	3	5

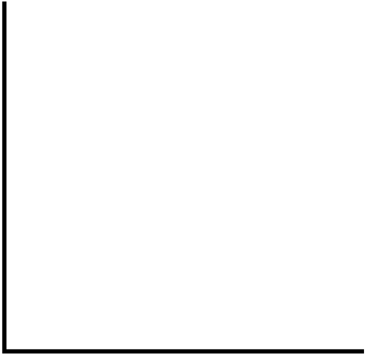
recording tableau Q(w)

w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)

Robinson-Schensted-Knuth algorithm



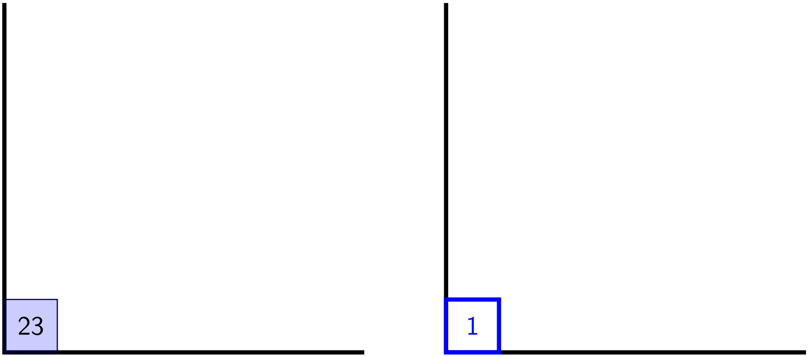
insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = \emptyset$

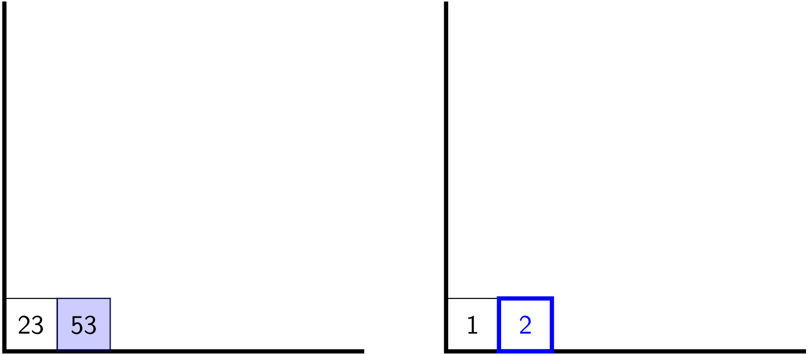
Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

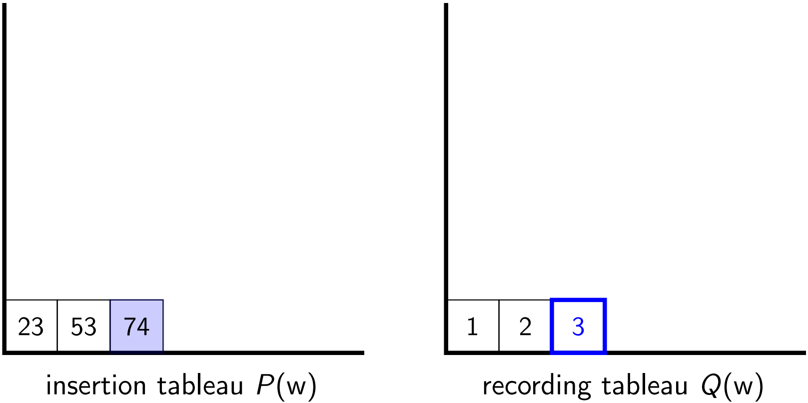
$\mathrm{w} = (23)$

Robinson-Schensted-Knuth algorithm



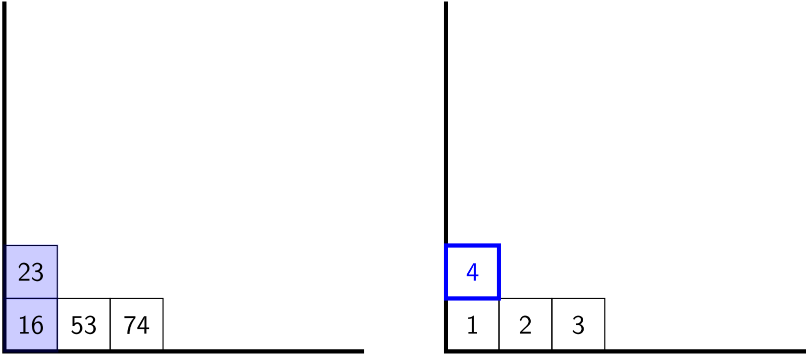
$\mathrm{w} = (23, 53)$

Robinson-Schensted-Knuth algorithm



$w = (23,\ 53,\ 74)$

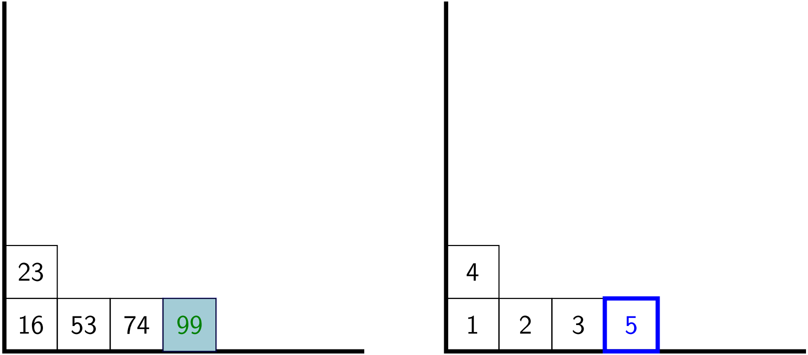
Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23,\ 53,\ 74,\ 16)$

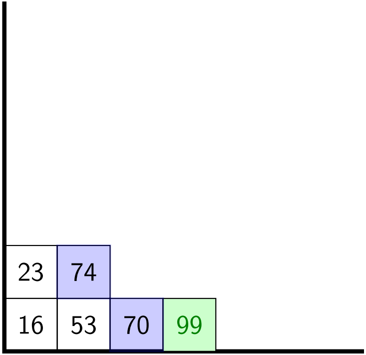
Robinson-Schensted-Knuth algorithm



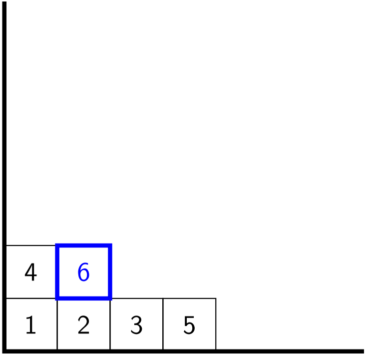
insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99)$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(w)$



recording tableau $Q(w)$

w = (23, 53, 74, 16, 99, 70)

Robinson-Schensted-Knuth algorithm

23	74	99	
16	53	70	82

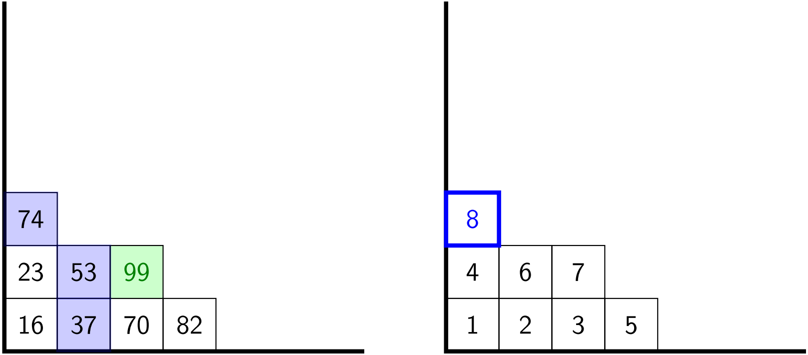
insertion tableau $P(\mathrm{w})$

4	6	7	
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82)$

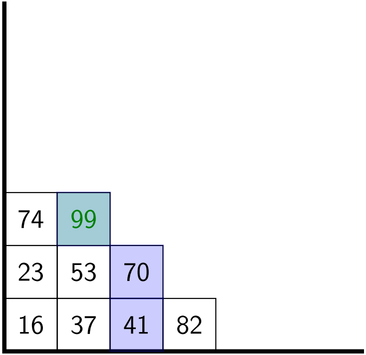
Robinson-Schensted-Knuth algorithm



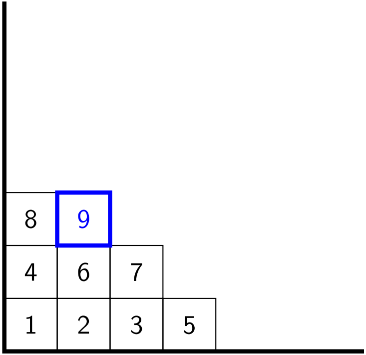
insertion tableau $P(w)$ recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37)$

Robinson-Schensted-Knuth algorithm



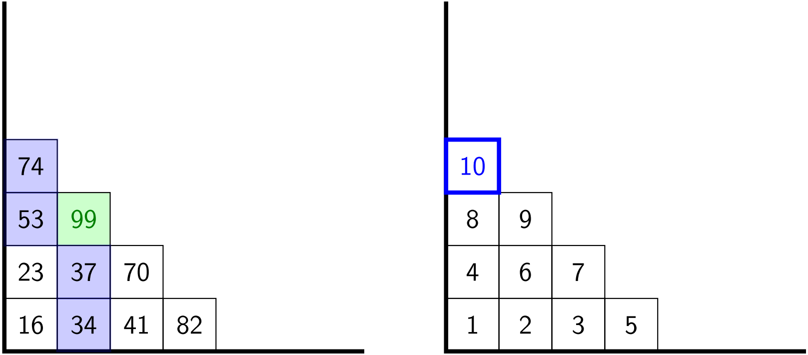
insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

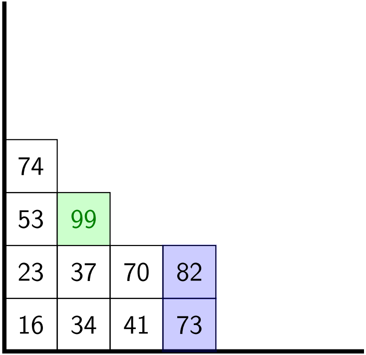
Robinson-Schensted-Knuth algorithm



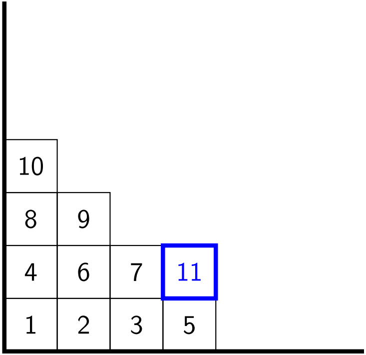
insertion tableau $P(w)$ recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34)$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(w)$



recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73)$

Robinson-Schensted-Knuth algorithm

74			
53			
23	99		
16	37	70	82
2	34	41	73

insertion tableau $P(w)$

12			
10			
8	9		
4	6	7	11
1	2	3	5

recording tableau $Q(w)$

$$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2)$$

Robinson-Schensted-Knuth algorithm

74			
53	99		
23	37		
16	34	70	82
2	24	41	73

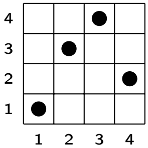
insertion tableau $P(\mathrm{w})$

12			
10	13		
8	9		
4	6	7	11
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2, 24)$

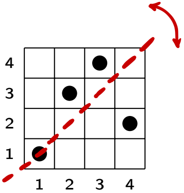
magic symmetries of RSK

$(1,3,4,2) =$ 

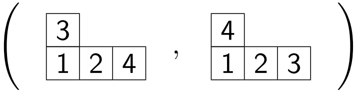
$$\mapsto \left(\begin{array}{|c|cc} \hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline \end{array} \; , \; \begin{array}{|c|cc} \hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right)$$

magic symmetries of RSK

$(1, 3, 4, 2) =$

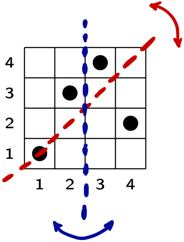


$\mapsto$



magic symmetries of RSK

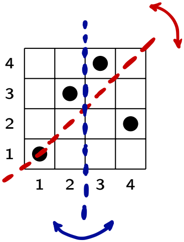
$(1, 3, 4, 2) =$



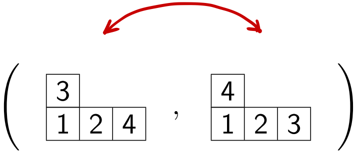
$$\mapsto \left(\begin{array}{|c|ccc} \hline 3 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \; , \; \begin{array}{|c|ccc} \hline 4 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \right)$$

magic symmetries of RSK

$(1,3,4,2) =$

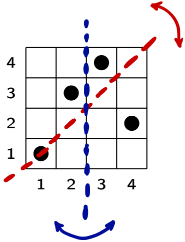


$\mapsto$

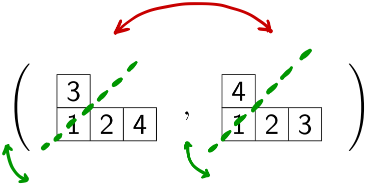


magic symmetries of RSK

$(1,3,4,2) =$



$\mapsto$



magic symmetries of RSK

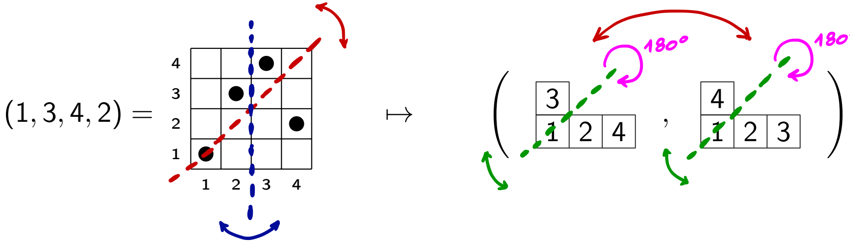
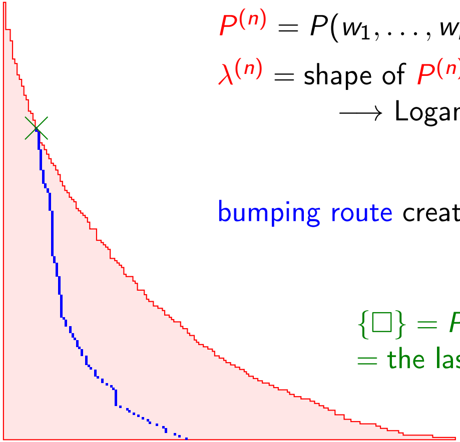


exhibit B: how it was created?

let $w_1, \dots, w_n$ be independent random variables
with the uniform distribution on $[0,1]$

$0 \leq s \leq 1$



$P^{(n)} = P(w_1, \dots, w_n)$

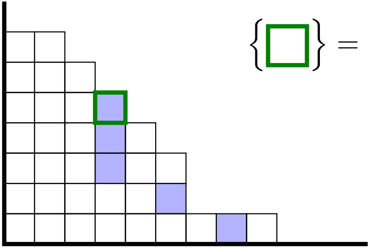
$\lambda^{(n)} =$ shape of $P^{(n)}$

$\longrightarrow$ Logan, Shepp, Vershik, Kerov (1977)

bumping route created in the insertion $P^{(n)} \leftarrow s$
(in this example $s = 0.2$)

$\{\Box\} = P(w_1, \dots, w_n, s) \setminus P(w_1, \dots, w_n)$
$=$ the last box in the bumping route

the end of the bumping route



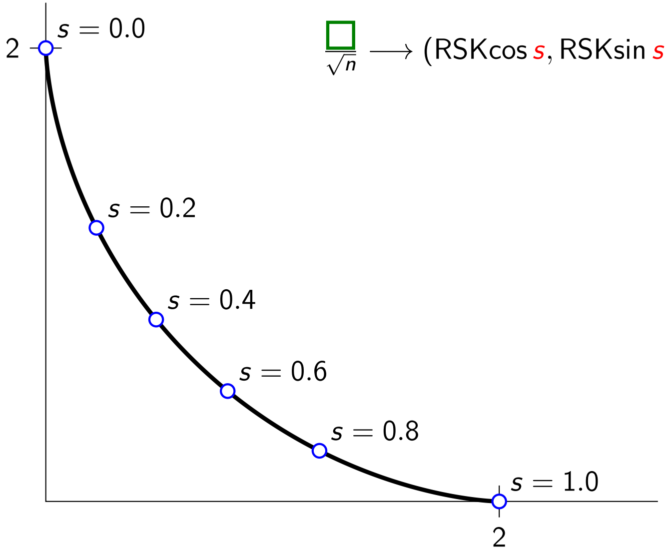
$$\{\square\} = P(w_1, \dots, w_n, s) \setminus P(w_1, \dots, w_n)$$

Theorem (Dan Romik, Piotr Śniady 2015)

$$\frac{\square}{\sqrt{n}} \xrightarrow[\text{in probability}]{n\to\infty} (\mathrm{RSKcos}\, s, \mathrm{RSKsin}\, s)$$

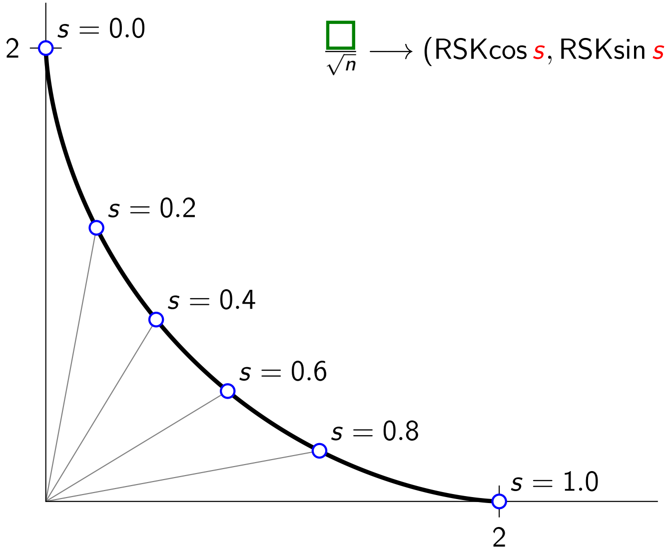
the end of the bumping route

$$\frac{\square}{\sqrt{n}} \longrightarrow (\mathrm{RSKcos}\, s, \mathrm{RSKsin}\, s)$$

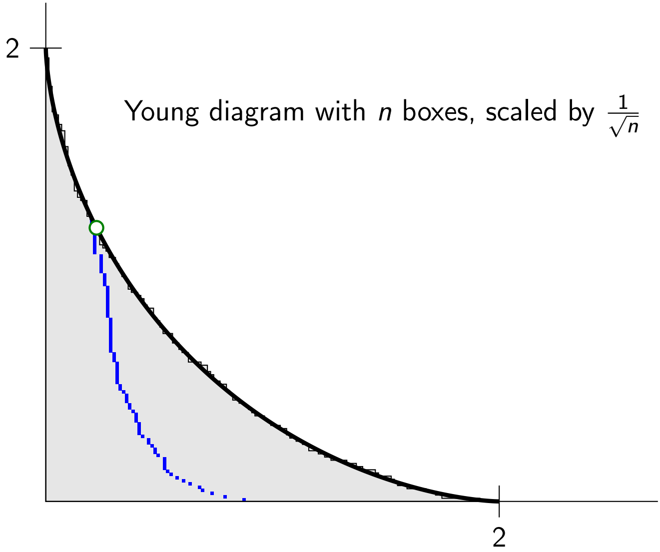


the end of the bumping route

$$\frac{\square}{\sqrt{n}} \longrightarrow (\mathrm{RSKcos}\, s, \mathrm{RSKsin}\, s)$$

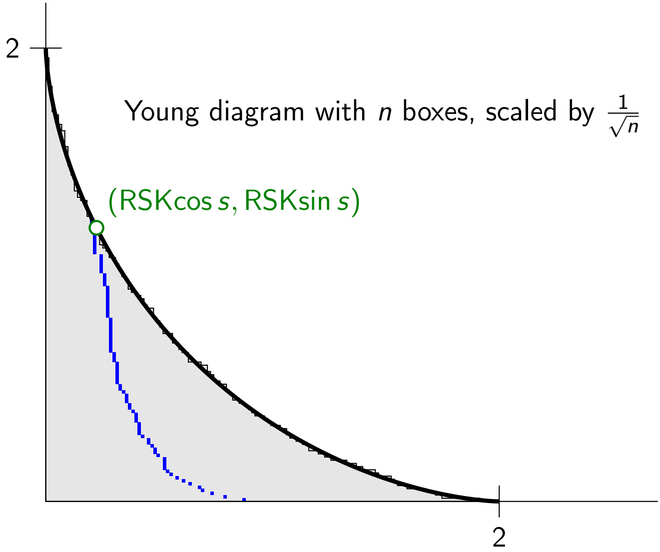


the bumping route



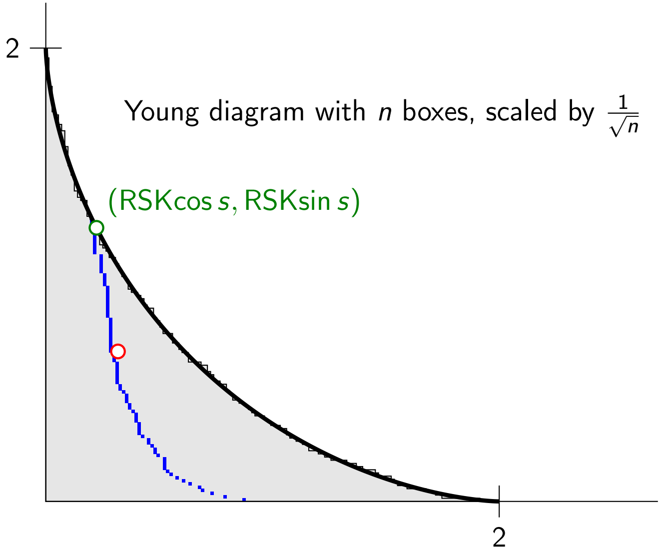
Dan Romik, Piotr Śniady 2016

the bumping route



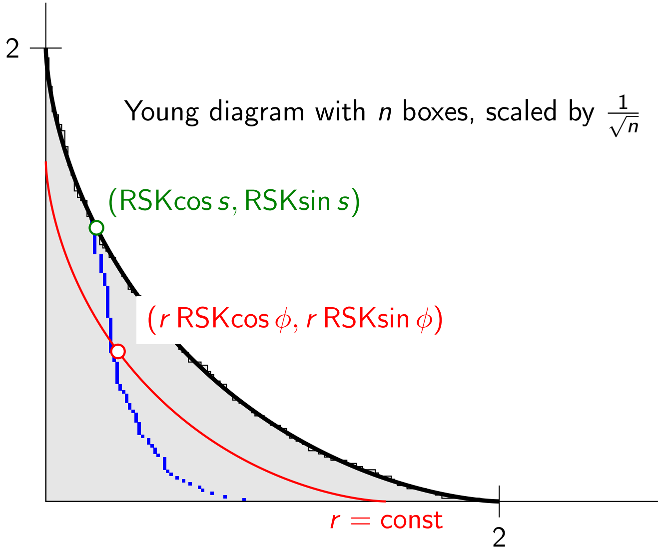
Dan Romik, Piotr Śniady 2016

the bumping route



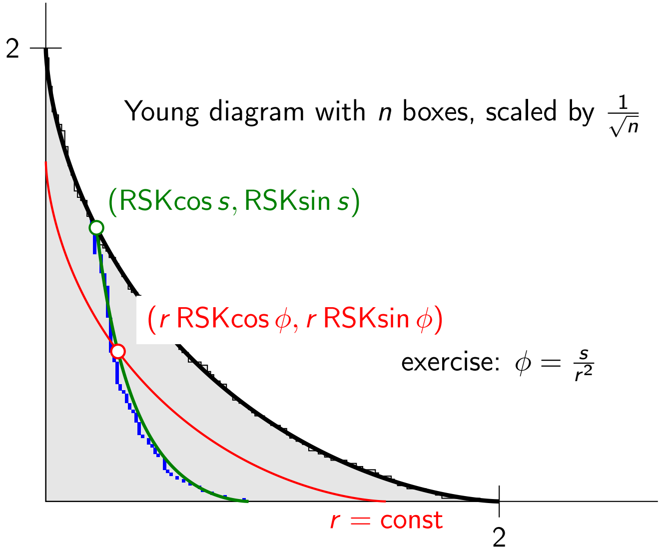
Dan Romik, Piotr Śniady 2016

the bumping route



Dan Romik, Piotr Śniady 2016

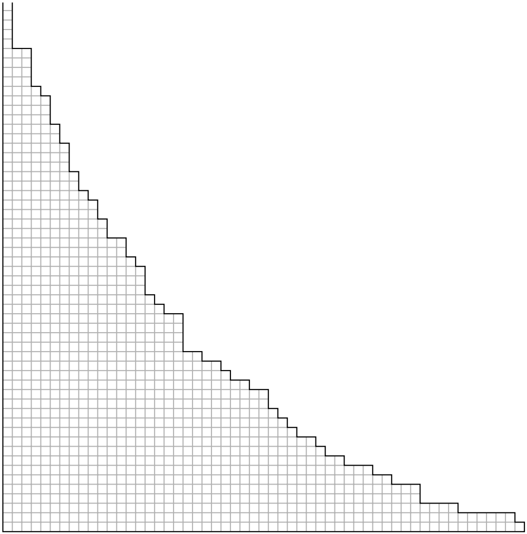
the bumping route



Dan Romik, Piotr Śniady 2016

diffusion of a box in the insertion tableau $P(\mathrm{w})$

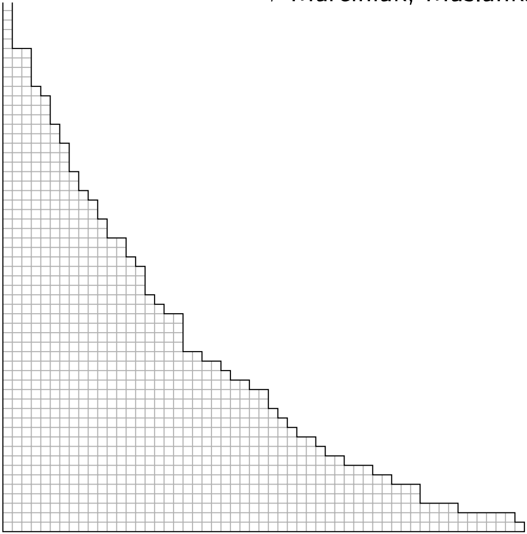
⟶ Mikołaj Marciniak (2022)



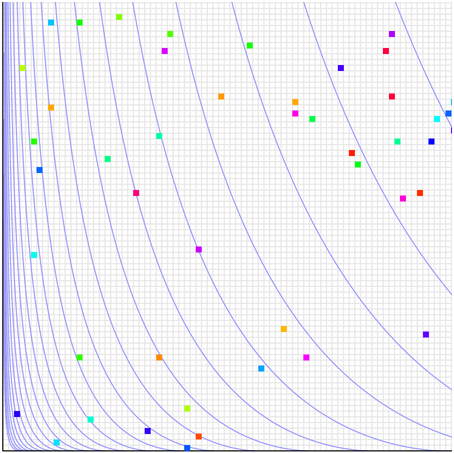
diffusion of a box in the insertion tableau $P(\mathrm{w})$

will this box ever reach the first column?

$\longrightarrow$ Marciniak, Maślanka, Śniady 2021



hydrodynamics of the insertion tableau $P(\mathrm{w})$



$m = 1.6 \cdot 10^4$

representation theory of S_n

- representation:

$$\rho\colon S_n \to \mathrm{End}(V)$$

  V is finite dimensional
- irreducible representations,
- irreducible characters,

repres. theory of $S_\infty = \bigcup_{n\geq 1} S_n$

- representation:

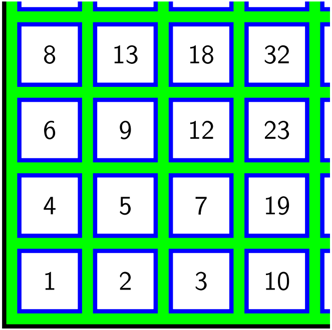
$$\rho\colon S_\infty \to B(\mathcal{H})$$

  $\mathcal{H}$ is a *Hilbert space*
- *factorial representations* $\longrightarrow$ operator algebras
- *extremal characters*,

Vershik, Kerov:
link between

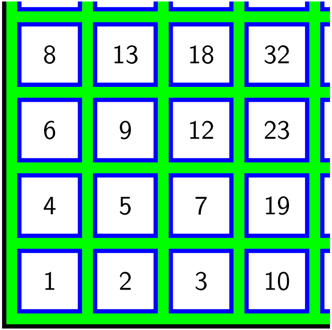
- factorial representations of S_∞,
- RSK applied to random input,
- random infinite tableaux,

infinite version of RSK



jeu de taquin

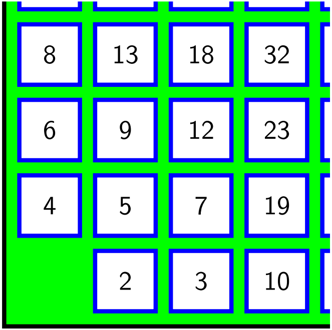
start with (infinite) tableau
$t = Q(w_1, w_2, \dots)$,



jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

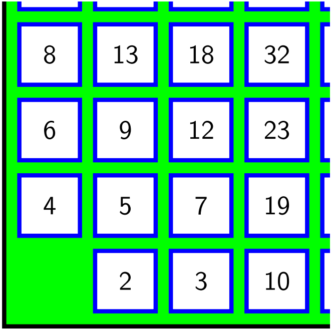
① remove corner box,



jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

① remove corner box,

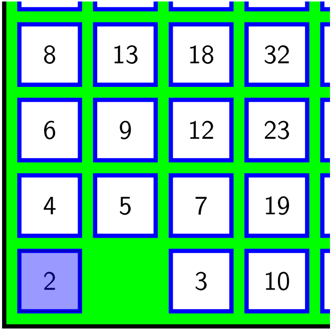


jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

① remove corner box,

② sliding,

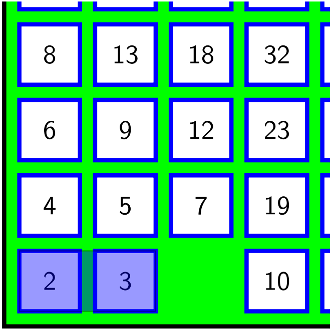


jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

① remove corner box,

② sliding,

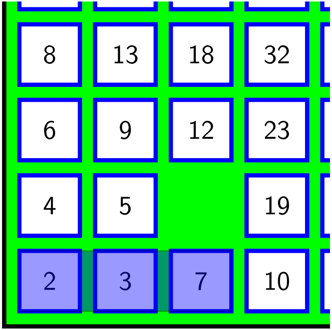


jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

① remove corner box,

② sliding,



jeu de taquin

start with (infinite) tableau
$t = Q(w_1, w_2, \dots)$,

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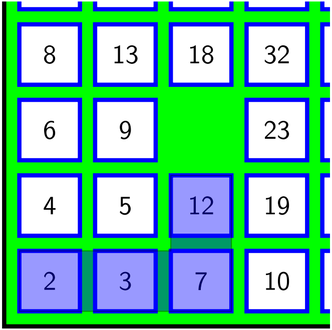

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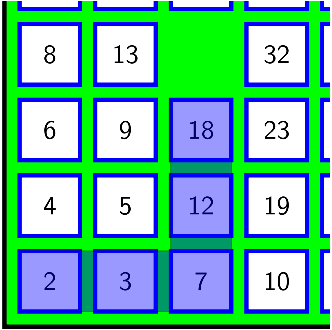

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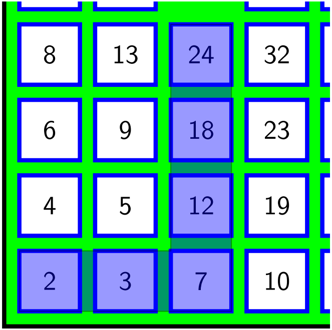

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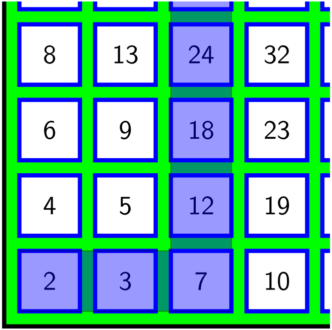

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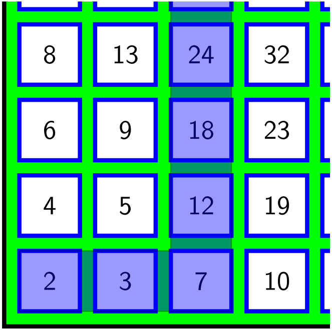

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start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

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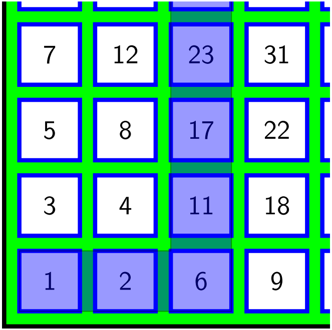

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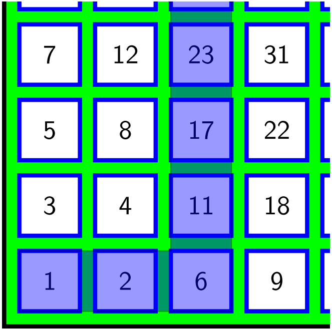

jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

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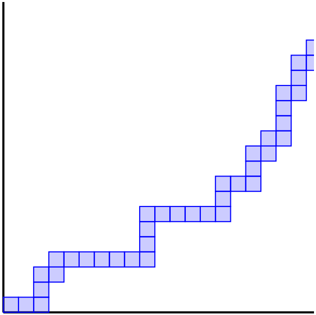
output:

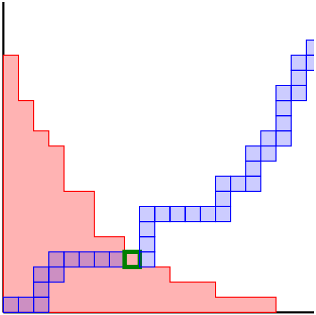
- new tableau $= Q(\cancel{w_1}, w_2, w_3, \dots)$,
- blue trajectory

trajectory of jeu de taquin has an asymptote



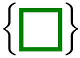
if $t = Q(w_1, w_2, \dots)$ is a random infinite tableau ...

trajectory of jeu de taquin has an asymptote



if $t = Q(w_1, w_2, \dots)$ is a random infinite tableau . . .

$\lambda^{(n)} = \{\text{boxes} \le n\}$



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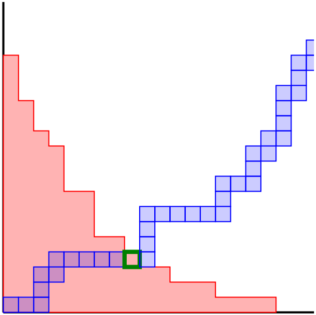

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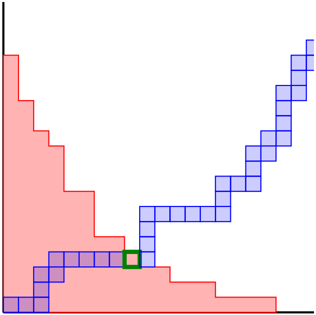

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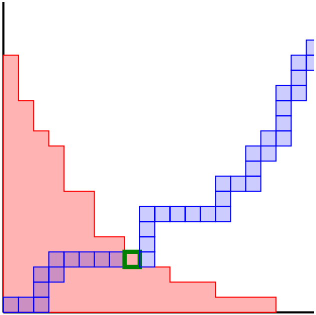
$$Q(1 - w_n, \dots, 1 - w_1) \setminus Q(1 - w_n, \dots, 1 - w_2)$$

trajectory of jeu de taquin has an asymptote



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$$Q(1-w_n, \dots, 1-w_1) \setminus Q(1-w_n, \dots, 1-w_2)$$

$$\approx \sqrt{n}\big(\text{RSKcos}(1-w_1), \text{RSKsin}(1-w_1)\big)$$

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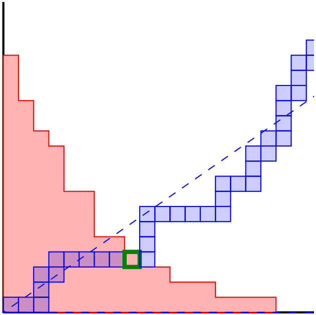

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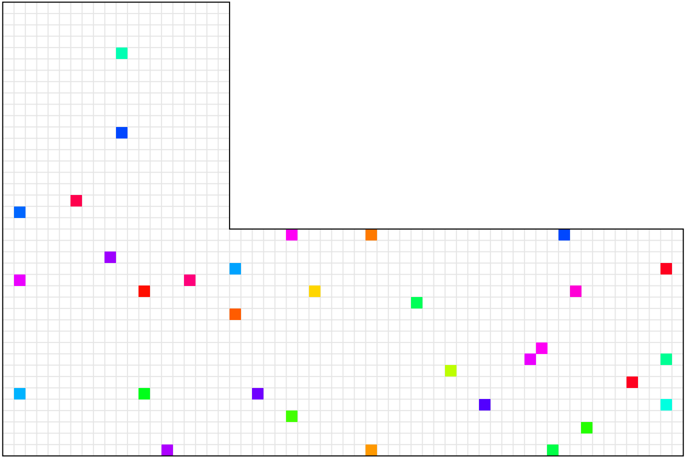
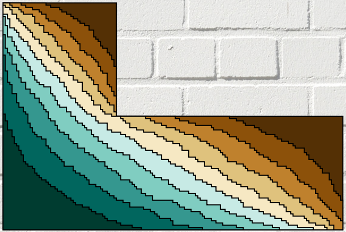
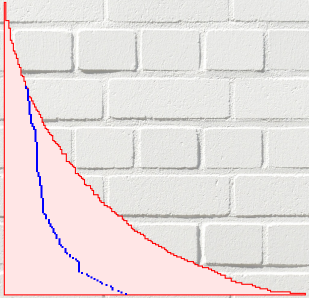
jeu de taquin in action



⟶ Łukasz Maślanka, Piotr Śniady 2022

conclusions





asymptotic / visual viewpoint may give new questions,

interesting from the algebraic combinatorics viewpoint

some coauthors

Mikołaj Marciniak

Łukasz Maślanka

transparencies, references, homework available on
`http://psniady.impan.pl/fpsac`